A note on grid transfer operators for multigrid methods [1]

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Abstract

The Local Fourier analysis (LFA) is a classic tool to prove convergence theorems for multigrid methods (MGMs). In particular, we are interested in optimality that is a convergence speed independent of the size of the involved matrices. For elliptic partial differential equations (PDEs), a well known optimality result requires that the sum of the orders of the grid transfer operators is not lower than the order of the PDE to solve. Analogously, when dealing with MGMs for Toeplitz matrices in the literature an optimality condition on the position and on the order of the zeros of the symbols of the grid transfer operators has been found. In this work we show that in the case of elliptic PDEs with constant coefficients, the two different approaches lead to an equivalent condition. We argue that the analysis for Toeplitz matrices is an algebraic generalization of the LFA, which allows to deal not only with differential problems but also for instance with integral problems. The equivalence of the two approaches gives the possibility of using grid transfer operators with different orders also for MGMs for Toeplitz matrices. We give also a class of grid transfer operators related to the B-spline's refinement equation and we study their geometric properties. This analysis suggests further links between wavelets and multigrid methods. A numerical experimentation confirms the correctness of the proposed analysis.

Key words: Multigrid methods, Toeplitz matrices, local Fourier analysis.
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1 Introduction

Multigrid methods (MGMs) are widely used for solving elliptic PDEs. The convergence analysis is usually done in the case of constant coefficients. Let us consider standard finite differences discretization for the following d-dimensional problem

$$\begin{cases} (-1)^q \sum_{i=1}^d \dfrac{\mathrm{d}^{2q}}{\mathrm{d}x_i^{2q}} u(x) = g(x), & x \in \Omega = (0,1)^d, \ q \geq 1, \\ \text{boundary conditions on } \partial\Omega, \end{cases} \tag{1}$$

where $x = (x_1, \ldots, x_d)$. For simplicity of the presentation we assume the same meshsize h for each dimension, but generalization to $h = (h_1, \ldots, h_d)$ is possible. Hence, approximating (1) by centered finite differences on a uniform grid of n subintervals of size h in each dimension, we obtain the linear system

$$A_n \mathbf{y} = \mathbf{b} \tag{2}$$

of order $n^d \times n^d$. Neglecting the boundary conditions, the matrix A_n is a d-level Toeplitz matrix and it is banded at each level.

Let L_h be a discretization of the differential operator in (1), then its Fourier transform is

$$\hat{L}(\omega) = \sum_{j \in \mathbb{Z}^d} l_j \, \mathrm{e}^{\mathrm{i}\langle jh | \omega \rangle}, \tag{3}$$

for $\omega \in [-\pi/h, \pi/h]^d$, $\mathrm{i}^2 = -1$, and

$$l_j = \frac{h^d}{(2\pi)^d} \int_{[-\pi/h, \pi/h]^d} \hat{L}(\omega) \, \mathrm{e}^{-\mathrm{i}\langle jh | \omega \rangle} \, \mathrm{d}\omega, \tag{4}$$

where the operations between multi-indices are intended component wise and $\langle \cdot | \cdot \rangle$ denotes the usual scalar product between vectors. Using centered finite differences of precision 2 and minimal bandwidth, the polynomial $\hat{L}(\omega)$ has degree $2q$, that is the order of the PDE (1), and L_h is completely defined from its d-dimensional stencil formed by the coefficients l_j, with $j_i = -2q + 1$, $\ldots$, $2q - 1$ for $i = 1, \ldots, d$.

It is well-known that MGMs are optimal solvers for PDEs of the form (1), i.e., they require about a constant number of iterations varying h and each iteration has an arithmetic cost proportional to the matrix-vector product [20]. Such property is obtained imposing a well known condition on the order of the grid transfer operators:

$$m_r + m_p \geq m, \tag{5}$$

where $m = 2q$ is the order of the PDE, m_r is the order of the restriction and m_p is the order of the prolongation [3], denoted as high frequencies order in [13,22]. The condition (5) follows from the LFA for the two grid method (TGM). In order to obtain more powerful grid transfer operators, that is in order to devise an optimal MGM, inequality (5) should to be satisfied strictly [20]. We note that the LFA does not consider the border effects, i.e., it assumes periodic boundary conditions or an infinite domain [3].

MGMs for multilevel positive definite Toeplitz matrices have been developed in the years looking only to the linear system (2), independently of the continuous problem [11,12,17,2,6,14]. A MGM for Toeplitz matrices was early defined in [11] using a powerful eigenvalue interlacing property with the matrices in the τ algebra (the class of matrices diagonalized by discrete sine transforms of type I). This first proposal was extended to the multilevel case in [12,6,17]. Since Toeplitz matrices do not define an algebra and hence are difficult to manipulate, convergence results are provided using matrix algebra approximations like τ or circulant matrices having the same spectral distribution of the Toeplitz matrices. In other words, we require that the circulant or the τ approximates share the same symbol of the original Toeplitz matrix [10,21]. In this paper we consider the circulant case. A MGM for circulant matrices was introduced in [19]. Furthermore in [2,1], generalizing the techniques used in [17], and using the Ruge and Stüben theory [16] and the Perron-Frobenius theorem, a complete proof of the optimality of the V-cycle for multilevel circulant and τ matrices was proposed. This analysis leads to a stronger condition with respect to the previous two grid analysis.

In this paper we show that the techniques used in [11,12,17,19] represent a linear algebra generalization of the LFA [3,13], in the case of the Galerkin approach. Indeed, they lead to a condition analogous to (5), but on the order of the zeros of the generating functions of the grid transfer operators. The letter represents a wide generalization since the case of discretization of elliptic PDEs of order $2q$ corresponds to the case of generating function which are nonnegative (ellipticity), with a unique zero at zero of order $2q$ (consistency condition). In other words, the case of discretization of elliptic PDEs is a subcase of nonnegative symbols with unique zero at zero, which in turn represents the case when the algebraic problem is ill-conditioned in a subspace of low frequencies. Therefore by using the Toeplitz approach other cases can be considered including the case when the ill-conditioning arises in high-frequencies: we recall that the latter characterizes some integral problem related to signal/images restoration. We will show that considering the problem (1), the LFA done in [3,13,22] and the Toeplitz approach [11,17] (which was introduced independently) are essentially equivalent. As already stressed, second approach is more general since it can be applied also when the zero of the symbol is not at the origin, or there exist several zeros (multiple sources of ill-conditioning). By the Galerkin approach, we have the only limitation that

the restriction must be proportional to the transpose of the prolongation, but in this paper we will show that in practical implementations this condition is not necessary. More precisely, we will generalize the MGM for Toeplitz matrices to the case of a restriction different to the transpose of the prolongation. A first suggestion to consider the linear algebra tools for Toeplitz matrices as a generalization of the LFA for multigrid methods was given in [15,18].

In this paper we also define a class of grid transfer operators that satisfy the conditions in [13] but that are not interpolating. More in detail the considered operators are defined looking for the smallest support of the symbol for a fixed order and are related to the refinement equation of B-spline. We will give a geometrical interpretation of the operator of order 4 and it will be compared with the cubic interpolation. These B-spline grid transfer operators allow us to discuss some relations between wavelets and multigrid methods. Eventually, a numerical experimentation validates our proposals.

The paper is organized as follows. Firstly in §2 we present the results in [13]. In §3 we describe the MGM defined in [11,12,17] for multilevel Toeplitz matrices using the zeros of the generating functions and we compare the two-grid analysis with the results in [13]. In §4 we generalize the TGM described in §3 to the case of restriction not necessary proportional to the transpose of the prolongation. In §5 we give a new class of grid transfer operators with minimum support for a fixed order and we show that such class is related to the B-spline. This allows, in §6, some observations about the relations between wavelet and multigrid methods. In §7 some numerical results validate the previous proposals both for Toeplitz non-differential problems and for PDEs with nonconstant coefficients. The final, §8 is devoted to some concluding remarks.

2   The low and high frequencies order analysis

We introduce a grid transfer operator that is not effective alone (it has order zero), but which is the basic tool for developing more powerful projectors. It is the classic *down-sampling* operator, called *elementary restriction* in [13] and *cutting matrix* in [11]. In the one-dimensional case, we set $n = n^{(0)} > n^{(1)} > \cdots > n^{(l)} > 0$, $l \in \mathbb{N}$, such that $n^{(i+1)} = (n^{(i)} - (n^{(i)} \bmod 2))/2$ and we define the down-sampling matrix $K_{n^{(i)}} \in \mathbb{R}^{n^{(i+1)} \times n^{(i)}}$ as

$$[K_{n^{(i)}}]_{j,k} = \begin{cases} 1 \text{ if } j = 2k - (n^{(i)} + 1) \bmod 2, \\ 0 \text{ otherwise}, \end{cases} \quad k = 1, \ldots, n^{(i+1)}. \quad (6)$$

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In the d-dimensional case the down-sampling matrix is defined by tensor product as $K_{n^{(i)}} = K_{n_1^{(i)}} \otimes K_{n_2^{(i)}} \otimes \cdots \otimes K_{n_d^{(i)}}$.

Higher order grid transfer operators are defined by convolution with the down-sampling operator. The prolongation is $P_{n^{(i)}}(p_i) = T_{n^{(i)}}(p_i)K_{n^{(i)}}^T$, while the restriction is $R_{n^{(i)}}(r_i) = K_{n^{(i)}}T_{n^{(i)}}(r_i)$ that usually is the transpose, up to a constant factor, of the prolongation. The matrix $T_n(t)$ is the d-level Toeplitz matrix generated by the function t. The Toeplitz matrices will be described in Section 3. The symbols r_i and p_i should be trigonometric polynomials of low order to maintain the computational cost of the matrix vector product proportional to $O(N(n^{(i)}))$.

Since more powerful grid transfer operators give a greater computational cost, it is important to find sufficient conditions such that we can decide for a fixed problem the cheapest grid transfer operators that allows to obtain an optimal MGM. This task can be done using the LFA [3,13] obtaining the condition (5).

We define the set of all corners of x as $\Omega(x) = \{\, y \,|\, y_j \in \{x_j,\, \pi + x_j\},\, j = 1, \ldots, d \}$. With the change of variable $x = \omega h$, the set of all frequencies on the fine grid that correspond to the frequency ω on the coarse grid is $\{z = y/h \mid y \in \Omega(x)\}$. Moreover, according to the terminology in [11] we define the set of the "mirror" points of x as $\mathcal{M}(x) = \Omega(x) \setminus \{x\}$. Since in the rest of the section we will consider only two grids, n will denote the fine grid. Moreover, for unifying the treatment, a generic grid transfer operator is denoted by $B_n(g)$ where g is multiplied by a factor 2^d when $B_n(g)$ is the prolongation, i.e., $B_n(g) = R_n(g)$ or $B_n(g) = P_n(2^d g)$.

Definition 1 *The* Low Frequency order (LF) *of a grid transfer operator* $B_n(g)$ *is the largest number $s \geq 0$ for which*

$$g(x) = 1 + O(|x|^s), \qquad for\ |x| \to 0.$$

Definition 2 *The* High Frequency order (HF) *of a grid transfer operator* $B_n(g)$ *is the largest number $s \geq 0$ for which*

$$g(y) = O(|x|^s), \qquad \forall y \in \mathcal{M}(x),\ \ for\ |x| \to 0.$$

For $x = \omega h$, $|x| \to 0$ means $h \to 0$ since ω is fixed. For $h = (h_1, \ldots, h_d)$ we can define $|x| = \max_{i=1,\ldots,d}(|x_i|)$.

For the grid transfer operators LF and HF are more general then classic interpolation order.

Proposition 1 ([13])*(i) If a restriction leaves all polynomials of degree $s-1$ invariant, then the LF of the operator is s.*

(ii) If a prolongation leaves all polynomial of degree $s - 1$ invariant, then both the LF and HF are at least s.

For instance the linear interpolation has LF = HF = 2, while the cubic interpolation has LF = HF = 4.

Furthermore, we can derive the condition (5) from the following

Proposition 2 ([13]) *Given a constant-coefficient, linear differential operator of order m, a necessary condition for non-increasing the high frequencies arising from a coarse grid correction with two grids it is*

$$\gamma_r + \gamma_p \geq m, \tag{7}$$

where γ_p and γ_r are the HF of the prolongation and of the restriction respectively.

From Proposition 1 part *(ii)* the condition (7) is a generalization of the analogous condition on the interpolation order. The LF is important for the restriction thanks to Proposition 1 part *(i)*, but it seems not necessary for the two grid analysis in Proposition 2. However, in [3] it is shown that for an efficient MGM a further condition is that both LF and HF are positive. This further request arises also from the Galerkin approach (see [22]) and is natural for obtaining an effective MGM.

Eventually, we note that, since the grid transfer operation has to be computationally cheap, the function g in Definitions 1 and 2 should be a trigonometric polynomial of low degree. Moreover, from Proposition 1 a good class of grid transfer operators should have at least LF> 0. Interpolating operators define a class with LF=HF. A further class of operators with a fixed HF and LF> 0 will be described in Section 5.

3   A MGM for Toeplitz matrices by generating functions

In this section we briefly introduce the MGM defined and analyzed in [11,12,6,7,17,19,2,1] for the multidimensional τ, circulant, Toeplitz and other matrix-algebras related to trigonometric transforms.

Toeplitz matrices arise from the discretization of convolution operators with a shift invariant kernel and hence not only from PDEs, but also from several other applications, e.g., image deblurring problems [2]. Toeplitz matrices are completely defined by the matrix size and the symbol also called generating function. Let f be a continuous function on $\mathbb{R}^d$ and having period 2π with

respect to each variable, the Fourier coefficients of f are defined as

$$a_j = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(x)e^{-i\langle j|x\rangle}\, dx, \qquad j \in \mathbb{Z}^d. \tag{8}$$

Remark 1 *With the change of variable $x = \omega h$, it holds $a_j = l_j$ and $f(x) = \hat{L}(\omega)$.*

From the coefficients $\{a_j\}$ one can build [21] the sequence $\{T_n(f)\}$ of multilevel Toeplitz matrices. Every matrix $T_n(f)$ is explicitly written as

$$T_n(f) = \sum_{|j_1|\leq n-1} \cdots \sum_{|j_d|\leq n-1} a_{(j_1,\ldots,j_d)} J_n^{[j_1]} \otimes \cdots \otimes J_n^{[j_d]}.$$

Here $\otimes$ denotes the usual tensor product and $J_n^{[j_i]} \in \mathbb{R}^{n\times n}$ is the matrix whose entry (s,t) equals 1 if $s - t = j_i$ and is 0 elsewhere, for $i = 1,\ldots,d$. Many structural and spectral properties of $T_n(f)$ derive from its generating function f. Indeed, if f is real valued, then $a_{-j} = \bar{a}_j$ for every j and the matrices $T_n(f)$ are Hermitian for every n; if f is also non-negative but not identically zero then $T_n(f)$ is positive definite.

Remark 2 *The main difference between $f(x)$ and $\hat{L}(\omega)$ is that ω denotes the frequency for the current discretization step h, information that seems to be lost in f, but that comes out from the matrix $T_n(f)$ regarding the current discretization ($h = 1/(n+1)$). For instance, let L_h be the three point discretization of the Laplacian: then $l_0 = 2/h^2$ and $l_{-1} = l_1 = -1/h^2$. On the other hand, in the algebraic approach for Toeplitz matrices the constant factor $1/h^2$ is moved to the right hand side (rhs) obtaining $a_0 = 2$ and $a_{-1} = a_1 = -1$. However the information of the order 2 of the Laplace operator is preserved since $f(x) = 2 - 2\cos(x)$ vanishes at the origin with order 2. More in general, discretizing (1) with finite centered differences of minimal precision and moving the coefficient $1/h^{2q}$ to the rhs, by consistency, the symbol $f(x)$ vanishes at the origin with order $2q$.*

Convergence results for MGMs for PDEs are usually obtained neglecting the boundary conditions. In a similar way, MGMs for Toeplitz matrices are defined starting from matrix algebras like τ or circulant. Imposing periodic boundary conditions in (1), the matrix A_n in (2) is circulant. Circulant matrices are simultaneously diagonalized by the Fourier transform $F_n = \frac{1}{\sqrt{n}}[e^{-ij y_i^{(n)}}]_{i,j}$, where $y_i^{(n)} = 2\pi i/n$, $i = 0,\ldots,n-1$. More precisely, the algebra of the circulant matrices can be formally defined as $\left\{A_n \mid A_n = F_n \cdot \text{diag}(\mathbf{d}) \cdot F_n^H,\ \mathbf{d} \in \mathbb{C}^n\right\}$ where the vector $\mathbf{d}$ of the eigenvalues is equal to $f(\mathbf{y}^{(n)})$, $\mathbf{y}^{(n)} = (y_0^{(n)},\ldots,y_{n-1}^{(n)})$, and a circulant matrix will be denoted by $\mathcal{C}_n(f)$. In the d-dimensional case the indices involved are multiindices, $F_n = F_{n_1} \otimes \cdots \otimes F_{n_d}$ has size $N(n)$ and $\mathbf{y}^{(n)} = \mathbf{y}^{(n_1)} \times \cdots \times \mathbf{y}^{(n_d)}$, where $\times$ denotes the cartesian product.

We do not consider boundary effects, thus we will discuss only the circulant case assuming periodic boundary conditions. In such case, in order to maintain the same circulant structure at each level, we have to start with $n = n^{(0)} = 2^a$, where $a \in \mathbb{N}^d$. Moreover the grid transfer operators are defined as $P_{n^{(i)}}(p_i) = C_{n^{(i)}}(p_i)K_{n^{(i)}}^T$ and $R_{n^{(i)}}(r_i) = K_{n^{(i)}}C_{n^{(i)}}(r_i)$. In our case $A_n = \mathcal{C}_n(f)$ is singular since f vanishes at the origin which is a grid point. However, without losing generality, we assume A_n nonsingular replacing f with its stabilized version that is by correcting A_n by adding a special rank-one matrix. This correction is not consider here since it does not imply particular assumptions but it leads only to unnecessary complications in the notation [2].

Using the Galerkin approach, we must have $R_n(r) = P_n(p)^H$, i.e., $r = p$, and $A_{n/2} = P_n(p)^H A_n P_n(p)$. Thanks to the structure of $P_n(p)$ we obtain that $A_{n/2}$ belongs again to the circulant algebra [19]. Thanks to the Ruge-Stüben theory [16], the TGM and the V-cycle convergence analysis can be split in two independent conditions, one on the smoother and the other on p_i, for $i = 0, \ldots, l - 1$, i.e., on the grid transfer operators.

Remark 3 *Several simple iterative methods, like relaxed Jacobi, satisfy the smoothing condition, therefore the main task is the study of the approximation condition for the grid transfer operators.*

In [19] the optimality of the TGM was proved for circulant matrices, under the following conditions on the grid transfer operators.

Proposition 3 ([19]) *Let the coefficient matrix be $A_n = \mathcal{C}_n(f)$ with f having a unique zero at x^0. Defining $P_n(p) = \mathcal{C}_n(p)K_n^T$ and $R_n(r) = \alpha P_n(p)^H$, i.e. $r = \alpha p$, $\alpha \in \mathbb{R}\backslash\{0\}$, where p is a trigonometric polynomial non identically zero and such that for each $x \in [-\pi, \pi)^d$*

$$\limsup_{x \to x^0} \left| \frac{p(y)^2}{f(x)} \right| = c < +\infty, \qquad \forall\, y \in \mathcal{M}(x), \tag{9a}$$

where

$$\sum_{y \in \Omega(x)} p(y)^2 > 0, \tag{9b}$$

then defining $A_{n/2} = \alpha P_n(p)^H A_n P_n(p)$ the TGM is optimal.

Proof. For $\alpha = 1$ see [19]. For $\alpha \neq 1$ it is enough to observe that the coarse grid correction $CGC = I - P_n(R_n A_n P_n)^{-1}R_n A_n$ is independent of α. $\qquad\square$

In order to compare this result with the Proposition 2 we have to require $p = 2^d r$, thus the (7) becomes $2\gamma_r \geq m$. We show the equivalence between two different convergence analysis for elliptic PDEs with constant coefficients: the LFA described in Section 2 and the analysis for Toeplitz matrices based on the zeros of the generating functions described here.

Proposition 4 *Let $P_n(p) = R_n(2^d r)^H$, discretizing (1) by finite centered differences of order 2 and minimal bandwidth, the conditions (7) and (9a) are equivalent.*

Proof. By Definition 2 $B_n(g)$ has HF= s if and only if $g(y) = 0$ with order s for all $y \in \mathcal{M}(x)$. The discretization of an elliptic constant coefficient PDE of order m by finite centered differences of precision 2 and minimal bandwidth leads to $A_n = \mathcal{C}_n(f)$ (in the case of periodic BCs) with f vanishing at the origin with order m (see Remark 1). From condition (9a) p (or equivalently r) must be chosen such that $p(y) = 0$, for all $y \in \mathcal{M}(0)$ with order $2\gamma_p \geq m$. This is exactly the inequality in (7). $\qquad\qquad\square$

The previous proposition shows that in the case of $p = 2^d r$ and using the Galerkin approach, condition (9a) is a generalization of condition (7) to general problems not necessarily of differential type. The main difference between the two approaches relies in the coarse strategy. The results in [13] and summarized in Section 2 assume a discretization of the same PDE with the same formula at each grid. This imposes a right scaling of the grid transfer operators (i.e., $B_n(g)$ has LF > 0 iff $g(0) = 1$). The latter is not necessary in the Galerkin approach adopted by the Toeplitz analysis, since the coarse matrix is defined as $A_{n/2} = \alpha P_n(p)^H A_n P_n(p)$. Indeed the condition (9b) requires only $p(0) \neq 0$. More specifically, p can be defined up to a scaling factor since this gives only a different scaling of $A_{n/2}$. However, the two approaches are comparable because from item 2 in Proposition 6, the coarse problem vanishes again at the origin and with the same order of the finer problem [17]. Using the PDE language, this means that for the Galerkin approach the linear system at the coarse grid is essentially (neglecting boundary conditions) the discretization of the same PDE with a formula of the same order.

At the end of Section 2 we noted that Proposition 2 does not requires any condition on the LF of the grid transfer operators. The only interest on the LF could be deduced from Proposition 1, and mainly for the restriction. On the other hand, the TGM condition (9b) requires that the grid transfer operators have a positive LF (up to a scaling factor). This is exactly the same requirement obtained in [22] for the Galerkin strategy and in [3] for an efficient MGM. In fact a condition LF= 0 is equivalent to violate (9b) which implies $p(x^0) = 0$. As a consequence, the associated grid transfer operators could fail to be full rank. The latter produces an increase of the ill-conditioning and could lead to singularity at the lower levels with a potential substantial change in the subspace related to small eigenvalues.

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4 MGM for Toeplitz matrices with a prolongation different from the transpose of the restriction

In practical implementation the condition

$$R_n(r) = \alpha P_n(p)^H \tag{10}$$

seems to be not necessary. The only request is that $A_{n/2} = R_n(r)A_nP_n(p)$ is again positive definite for a recursive application of the algorithm. On the other hand, the condition (10) is very useful for a theoretical analysis, because if $r \neq p$ the coarse grid correction $CGC = I_n - P_n(p)A_{n/2}^{-1}R_n(r)A_n$ is again a projector, but it is not longer unitary with respect to the scalar product $<\mathbf{y}, \mathbf{x}>_{A_n} = \mathbf{y}^H A_n \mathbf{x}$, A_n Hermitian positive definite, for all $\mathbf{y}, \mathbf{x} \in \mathbb{C}^n$. For the well definiteness of a MGM, mainly to ensure that the same smoother is convergent also to the coarse levels, $A_{n/2}$ should be positive definite to apply recursively the algorithm. This condition is easy satisfied for $p_i \geqslant 0$, $r_i \geqslant 0$ (not identically zero) and either both even or both odd, $i = 0, \ldots, l-1$. More generally we could use $p_i r_i \geq 0$ with isolated zeros, $i = 0, \ldots, l-1$. Therefore, the following generalization of Proposition 3 can be conjectured.

TGM CONDITIONS. *Let the coefficient matrix be $A_n = \mathcal{C}_n(f)$, with f having a unique zero at x^0. Defining $R_n = K_n\mathcal{C}_n(r)$ and $P_n = \mathcal{C}_n(p)K_n^T$ where p and r are trigonometric polynomials non identically zero and such that for each $x \in [-\pi, \pi)^d$*

$$\limsup_{x \to x^0} \left| \frac{r(y)p(y)}{f(x)} \right| = c < +\infty, \qquad \forall\, y \in \mathcal{M}(x), \tag{11a}$$

where

$$\sum_{y \in \Omega(x)} r(y)p(y) \neq 0. \tag{11b}$$

Then, defining $A_{n/2} = R_n(r)A_nP_n(p)$, the TGM is optimal.

These two conditions are motivated by the analysis in the previous section and by the following Proposition 5 that extends Proposition 4 to the case of $r \neq \alpha p$. Moreover, the numerical experiments in Section 7 will validate these conditions.

Proposition 5 *Discretizing (1) by finite centered differences of order 2 and minimal bandwidth, the conditions (7) and (11a) are equivalent.*

Proof. The proof is analogous to that of Proposition 4. It is enough to observe that if r and p vanish at y with order γ_r and γ_p respectively, then rp vanishes at y with order $\gamma_r + \gamma_p$. $\qquad\square$

We provide a further result useful to implement the corresponding MGM.

Proposition 6 *Let $A_n = \mathcal{C}_n(f)$, $P_n(p) = \mathcal{C}_n(p)K_n^T$, $R_n(r) = K_n\mathcal{C}_n(r)$, with f, p, r trigonometric polynomials and p, r satisfying conditions* (11). *Then*

(1) $A_{n/2} = R_n(r)A_nP_n(p)$ *coincides with* $\mathcal{C}_{n/2}(\hat{f})$ *where*

$$\hat{f}(x) = \frac{1}{2^d} \sum_{y\in\Omega(x/2)} r(y)f(y)p(y), \qquad x \in [-\pi, \pi)^d. \qquad (12)$$

(2) If $x^0 \in [-\pi, \pi)^d$ is a zero of f, then $y^0 = 2x^0 \bmod 2\pi$ is a zero of $\hat{f}$. Moreover the order of the zero y^0 of $\hat{f}$ is exactly the same as the one of the zero x^0 of f.

Proof. The essentials of the proof in the case of $r = p$ can be found in [19]. For $r \neq p$ we can proceed similarly. We sketch the main steps for the one dimensional case and, at the end, we extend it to the multidimensional case, mainly for emphasizing the algebraic interpretation of the frequencies packaging used in the LFA.

The main relationship is

$$K_nF_n = \frac{1}{\sqrt{2}}\left[F_{n/2} \mid F_{n/2}\right],$$

that implies

$$A_{n/2} = K_n\mathcal{C}_n(r)\mathcal{C}_n(f)\mathcal{C}_n(p)K_n^T$$

$$= \frac{1}{2}\left[F_{n/2} \mid F_{n/2}\right]\mathrm{diag}_{j=0,\ldots,n-1}\left(rfp\left(\frac{2\pi j}{n}\right)\right)\begin{bmatrix} F_{n/2}^H \\ F_{n/2}^H \end{bmatrix}$$

$$= \frac{1}{2}F_{n/2}\mathrm{diag}_{j=0,\ldots,n/2-1}\left(rfp\left(\frac{\pi j}{n/2}\right) + rfp\left(\frac{\pi j}{n/2} + \pi\right)\right)F_{n/2}^H.$$

that is the (12) for $d = 1$.

In the multidimensional case $K_nF_n = 2^{-d/2}F_{n/2}G_n$, where $F_n = \bigotimes_{j=1}^d F_{n_j/2}$ and $G_n = \bigotimes_{j=1}^d \left([1\ 1] \otimes I_{n_j/2}\right)$. Therefore $G_nD_n(rfp)G_n^T = D_{n/2}(2^d\hat{f})$, where $D_k(h) = \mathrm{diag}_{0\leqslant j\leqslant k-e}(h(2\pi j/n))$ and $\hat{f}$ is defined by (12).

The claim in item 2 is a consequence of item 1 and of relations (11). $\qquad\square$

Thanks to Proposition 5, the TGM conditions in (11) give a complete generalization of Proposition 2, also for non-differential problems since the case of generating functions vanishing at points different from the origin is also included. Therefore the analysis based on the zeros of the generating function can be considered an algebraic generalization of the LFA also to non-

differential problems. For instance, some discretized integral problems have a generating function vanishing at πe, $e = (1, \ldots, 1)^T \in \mathbb{R}^d$, or more generally at some πx, $x_i \in \{0, 1\}$, $i = 1, \ldots, d$, $||x||_\infty = 1$, with order $2q$. In this case $\mu_q(x) = 2^{-dq} \prod_{j=1}^d (1 - \mathrm{e}^{-\mathrm{i}x_j})^{\lfloor \frac{q}{2} \rfloor} \mathrm{e}^{\mathrm{i}x_j \lfloor \frac{q}{2} \rfloor}$ satisfies the conditions (11) and therefore it defines an optimal TGM. We note that $R_n = K_n T_n(\mu_q)$ is an high-pass filter and then it projects into the high frequencies. However, thanks to Proposition 6, the zero at the next level moves to the origin and at the coarser grids the problem becomes spectrally equivalent to the discretization of a constant coefficient elliptic PDE.

Finally, we recall the V-cycle optimality conditions for Toeplitz matrices given in [2] for $d = 1$ and in [1] for $d > 1$.

Proposition 7 ([2,1]) *Let $A_{n^{(i)}} = \mathcal{C}_{n^{(i)}}(f_i)$ be the coefficient matrix at the level i, for $i = 0, \ldots, l$, with f_i having a unique zero at x_i^0. Defining $A_{n^{(i+1)}} = R_{n^{(i)}}(r_i) A_{n^{(i)}} P_{n^{(i)}}(p_i)$, $P_{n^{(i)}}(p_i) = \mathcal{C}_{n^{(i)}}(p_i) K_n^T$ and $R_{n^{(i)}}(r_i) = \alpha P_{n^{(i)}}(p_i)^H$, i.e. $r_i = \alpha p_i$, $\alpha \in \mathbb{R} \backslash \{0\}$, where p_i is a trigonometric polynomial non identically zero and such that for each $x \in [-\pi, \pi)^d$*

$$\limsup_{x \to x^0} \left| \frac{p_i(y)}{f_i(x)} \right| = c < +\infty, \qquad \forall y \in \mathcal{M}(x), \tag{13a}$$

where

$$\sum_{y \in \Omega(x)} p_i(y)^2 > 0. \tag{13b}$$

Then the V-cycle is optimal.

We observe that (13a) defines a stronger condition on the order of the grid transfer operators with respect to the condition (9a). On the other hand, choosing $c = 0$ in (9a) or in (11a), which is equivalent to require that the (7) is satisfied strictly, is usually enough to obtain an optimal V-cycle as numerically shown in [19,20,2]. Following the same analysis done for Proposition (3), the Proposition (7) could be generalized to the case of $r \neq p$ and also applied to non-constant coefficients PDEs. In this last case, condition (13a) could be rewrite as $\gamma_r + \gamma_p \geq 2m$.

5   B-spline grid transfer operators

From the discussion at the end of Section 2, the HF is more important than LF: in fact, for the latter it is enough require LF> 0. A class of grid transfer operators having HF$= m$ and LF> 0 can be defined by

$$\varphi_m(x) = \prod_{j=1}^d \left(\frac{1 + \mathrm{e}^{-\mathrm{i}x_j}}{2} \right)^m. \tag{14}$$

Therefore every grid transfer operator with HF$= m$ has a generating function of the form $\varphi_m(x)\nu_m(x)$ such that $\nu_m(x) \neq 0$ for all $x \in \mathcal{M}(0)$ and $\nu_m(0) = 1$. With $\nu_m \equiv 1$ we obtain a class of projectors with minimal support for a fixed order m. We note that $\sqrt{2}\varphi_m$ is the symbol of the B-spline of order m in the multiresolution analysis (MRA) [8]. This technique is based on hierarchies of nested spaces $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$, defined through a basis generated (for $d = 1$) by translations and dilations $\beta(2^j x - k)$, $k \in \mathbb{Z}$, of a single scaling function β. A scaling function satisfies an equation of the type $\beta(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k \beta(2x - k)$, which expresses the nestedness of the spaces $V_j = \text{span}\{\beta(2^j x - k), k \in \mathbb{Z}\}$. Let $\hat{\beta}(v)$ be the Fourier transform of $\beta(x)$. Then

$$\hat{\beta}(v) = H(v/2)\hat{\beta}(v/2), \tag{15}$$

with $H(v) = 1/\sqrt{2}\sum_{k\in\mathbb{Z}} h_k\, \mathrm{e}^{-\mathrm{i}kv}$. The function $H(v)$ is called the symbol of β. A common application of the MRA is to approximate a high resolution $f \in V_j$ by a coarser function $\tilde{f} \in V_k$ with $k < j$ without losing a lot of information.

In the one dimensional case a simple scaling function is the Haar-function $\beta(x) = 1$ for $x \in [0,1)$ and zero otherwise, which satisfies the refinement equation $S_1(x) = S_1(2x) + S_1(2x - 1)$. The Haar-function S_1 is the simplest B-spline of order $m = 1$, and $\{S_1(x - k) : k \in Z\}$ is an orthonormal basis of $L_2(\mathbb{R})$. The B-spline of order m can be defined by $S_m = S_1 * S_{m-1}$, where $*$ is the convolution operator. For instance $S_2 = S_1 * S_1$ is such that $S_2(x) = 1/2(S_2(2x) + 2S_2(2x - 1) + S_2(2x - 2))$ and its translated $\tilde{S}_2(x) := S_2(x + 1)$, $\tilde{S}_2(x) = 1 - |x|$ for $x \in [-1, 1]$ and zero otherwise, is known as the hat function. We remark that $S_2 = 2\varphi_2$ and more generally $S_m = 2\varphi_m$.

The functions S_m are not centered, but they can be easy centered as previously done for S_2: instead of $\nu_m \equiv 1$ we take the shift $\nu_m = \mathrm{e}^{\mathrm{i}x\lfloor \frac{m}{2}\rfloor}$. In this way we define a class of centered projectors such that the symbol of order m is

$$\phi_m(x) = \prod_{j=1}^{d}\left(\frac{1 + \mathrm{e}^{-\mathrm{i}x_j}}{2}\right)^m \mathrm{e}^{\mathrm{i}x\lfloor \frac{m}{2}\rfloor}. \tag{16}$$

The ϕ_m have HF$= m$ and LF$= 2$. As previously observed, we note that a good class of grid transfer operators should have a high HF, while for the LF the only request is LF > 0. For $d = 1$, the ϕ_m, can be obtained using the Tartaglia's triangle as in Table 1. For $d = 2$ we take the tensor product of the one dimensional stencil and so on for $d > 2$.

The class of projectors defined in (16) is a scaled generalization of $p_{2k}(x) = \sqrt{2}(1 + \cos(x))^k$ proposed in [2] for even functions (zeros of order $m = 2k$) in the one dimensional case, indeed $p_{2k} = 2^{1/2-k}\phi_{2k}$. However, scaling factors does not change the effectiveness of the projector for the Galerkin approach. We remark that if m is odd than the grid transfer operators related to ϕ_m are not symmetric for vertex centered discretization, while they are symmetric for

Table 1
The refinement coefficients $h_k \neq 0$, $k \in \mathbb{Z}$ for $2^{m-\frac{1}{2}}\phi_m$ in the one dimensional case.

m	h_{-2}	h_{-1}	h_0	h_1	h_2
1			1	1	
2			1	2	1
3	1		3	3	1
4	1		4	6	4	1

cell centered discretization [22]. For instance [1 3 3 1] is the linear interpolation for cell centered discretization.

We consider for simplicity the one dimensional case, but the following observations are true also for $d > 1$. For vertex centered discretizations, we are interested in φ_{2k}. It is easy to prove that $P_n(\phi_2)$ is the linear interpolation. Which is the geometrical meaning of $P_n(\phi_4)$? Which are the relations between $P_n(\phi_4)$ and $P_n(g_c)$? Answers to these questions will be given in the next subsection.

5.1   A quadratic prolongation of order 4

In this subsection we give a geometric interpretation of $P_n(\phi_4)$ which has HF= 4 like the cubic interpolation.

The simplest but useless prolongation is

(**A**) $$p(x) \equiv 1.$$

Without losing in generality we consider n odd. For $\mathbf{y} \in \mathbb{C}^n$ and $\mathbf{x} \in \mathbb{C}^{\frac{n-1}{2}}$, $\mathbf{y} = P_n(1)\mathbf{x} = K_n\mathbf{x}$ does not reconstruct constant functions not identically zero. Particularly, the choice (**A**) does not provide a good approximation for the odd components. Therefore, in the standard MGM, $P_n(p)$ is frequently chosen as the linear interpolation

(**B**) $$p(x) = 1 + \cos(x).$$

Remark 4 *The choice (**B**) compared to the choice (**A**) leaves unchanged the even components but reinforces the odd components, which are not well approximated by the choice (**A**), with a linear interpolation.*

When the choice (**B**) is ineffective, it is usually replaced with the cubic interpolation. An alternative is given by $P_n(\phi_4)$ which has HF= 4 like the cubic interpolation but a smaller support. This prolongation follows a strategy similar to that used for deriving choice (**B**) from (**A**): it leaves the linear interpolation

for the odd components and reinforces the even components. From (16)

$$\phi_4(x) = 4^{-d} \prod_{j=1}^{d} \left(1 + \cos(x_j)\right)^2.$$

Therefore, in the one dimensional case, $r(x) = (1 + \cos(x))^2/4$ and

(**C**) $$p(x) = (1 + \cos(x))^2/2.$$

Remark 5 *With respect to the choice (**B**), this choice leaves unchanged the odd components but reinforces the even ones with a quadratic approximation:*

$$\mathbf{y}_j = \begin{cases} (\mathbf{x}_k + \mathbf{x}_{k+1})/2, & j = 2k+1, \\ (\mathbf{x}_{k-1} + 6\mathbf{x}_k + \mathbf{x}_{k+1})/8, & j = 2k, \end{cases} \qquad k = 1, \ldots, n, \qquad (17)$$

where we assume $\mathbf{x}_0 = \mathbf{x}_{n+1} = 0$.

The approximation for the even components of $\mathbf{y}$ is obtained taking the middle value of a *quadratic rational Bezier curve* defined from the three points $\{\mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{x}_{k+1}\}$. The Bernstein polynomial of order n is defined as

$$B_i^{(n)}(t) = \binom{n}{i} (1-t)^i t^{n-i}, \qquad t \in [0,1], \quad i = 1, \ldots, n.$$

A quadratic rational Bezier curve has the expression

$$C(t) = \frac{\sum_{i=0}^{2} \omega_i \mathbf{b}_i B_i^{(2)}(t)}{\sum_{i=0}^{2} \omega_i B_i^{(2)}(t)},$$

where $\mathbf{b}_i$ are the control points and ω_i are the associated weights for $i = 0, 1, 2$. Let $\mathbf{b}_i = \mathbf{x}_{k+i-1}$ for $i = 0, 1, 2$, $\omega_1 = 3/2$ and $\omega_0 = \omega_2 = 1/2$, then $C(\frac{1}{2}) = (\mathbf{x}_{k-1} + 6\mathbf{x}_k + \mathbf{x}_{k+1})/8$ which is the same of (17). In Figure 1 the previous quadratic approximation is shown for the computation of $\mathbf{y}_j$ with $j = 2k$. Furthermore in Figure 2 we compare the values obtained in the finer grid with the choice (**B**) (linear interpolation) and with the choice (**C**) (quadratic approximation).

We note a different philosophy between the cubic interpolation and this quadratic approximation. The cubic interpolation reconstructs exactly the polynomial of degree at most three, while the quadratic approximation does not. However, for the even nodes the cubic interpolation preserves the exact value as in the coarse grid. This can be useful for the TGM or when the coarse solution is well approximated. But when this is not the case, like for the V-cycle in some applications, it could be better to take an approximation that is an average at the neighboring nodes. The underling idea is that it is not useful to take
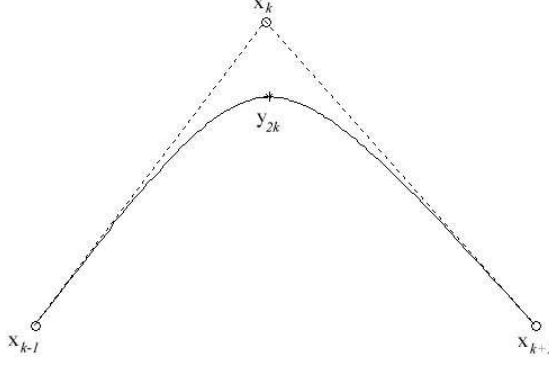
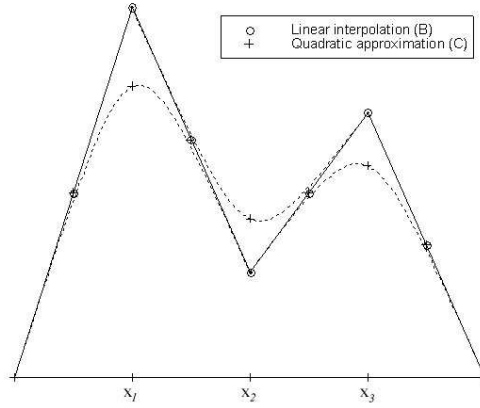Fig. 1. Even components in the finer grid computed with the choice (**C**).

Fig. 2. Computation of the points y_i for $i = 1, \ldots, 5$ in the finer grid using the linear interpolation (line) and the quadratic approximation (dotted).

a powerful prolongation and a poor restriction, because in this case we interpolate the solution of a coarse problem that does not represent well the finer problem.

The generating function of the cubic interpolation is $g_c(x) = \varphi_4(x)\nu(x)$, with $\nu(x) = 2 - \cos(x)$. The stencil of $B_n(g_c)$ is $\frac{1}{32}[-1 \quad 0 \quad 9 \quad 16 \quad 9 \quad 0 \quad -1]$ and the one of $B_n(\phi_4)$ is $\frac{1}{16}[1 \quad 4 \quad 6 \quad 4 \quad 1]$. In addition to the different philosophy previously emphasized, we observe the following mainly differences.

Remark 6 $B_n(g_c)$ has HF=LF=4, while $B_n(\phi_4)$ has HF=4 and LF=2.

Remark 7 *From a computational point of view, the two stencils have the same number of nonzero elements. Hence they have the same computational cost for the projection of a vector between coarse to fine or fine to coarse grids. The main difference is that the stencil of the cubic interpolation has a lager support. This implies that, mainly for $d > 1$, $B_n(\phi_4)$ defines a MGM that is computationally more efficient at the coarser grids. Indeed, using the Galerkin approach, the coarse matrix has a lower bandwidth. Moreover, $B_n(\phi_4)$ requires*

less boundary points, property that is very useful for a parallel implementation. In this way less communications are required among the nodes and we can employ a colored Gauss-Seidel with a smaller number of colors, increasing the parallelism degree.

From the previous remarks, as it will be confirmed by the numerical experimentation in Section 7, the cubic interpolation usually converges within less iterations with respect to the choice ($\mathbf{C}$) (see Remark 6), but both have the same asymptotic behavior since they have the same HF. Therefore, thanks to Remark 7, $B_n(\phi_4)$ could be a good alternative to $B_n(g_c)$, mainly for parallel implementations or V-cycle MGMs.

6 A comparison between MGMs and wavelets methods

In this subsection we recall some approximation properties of the ϕ_m defined in (16), or equivalently of the φ_m in (14). Moreover we discuss some relations between wavelets and multigrid methods.

Unfortunately $\{S_m(x - k) : k \in \mathbb{Z}, m > 1\}$ is not an orthogonal system. However, the S_m satisfy the quasi-interpolant Strang-Fix condition [8]:

$$\left(\frac{\partial}{\partial v}\right)^s \hat{S}_m(2k\pi) = 0, \qquad k \in \mathbb{Z}\backslash\{0\}, \quad s = 0, \dots, m - 1. \qquad (18)$$

It follows that the polynomials of degree at most $m - 1$ are contained in the space $V_0 = \text{span}\{S_m(x - k) : k \in \mathbb{Z}\}$. The same property is also usually expressed in terms of vanishing moments. Let ψ_m be the wavelet associated to S_m. The first $m - 1$ moments of ψ_m vanish, i.e.

$$\int_{-\infty}^{+\infty} x^s \psi_m(x) = 0, \qquad s = 1, \dots, m - 1. \qquad (19)$$

Starting from the orthogonality condition

$$|H(v)|^2 + |H(v + \pi)|^2 = 1, \qquad (20)$$

where $H(v)$ is the symbol defined in (15), and imposing the vanishing of the moments, Daubechies defined orthogonal wavelets [9]. The moment condition (19) is equivalent to require $H(x) = \varphi_m(x)\nu(x)$, such that $\nu(x)$ is a trigonometric polynomial with $\nu(0) = 1$. This means that orthogonal wavelets are obtained imposing orthogonality to S_m. For instance, for $m = 2$ we obtain the Daubechies wavelet of order 2 with scaling coefficients $h = \frac{1}{4\sqrt{2}}[1 + \sqrt{3}, \quad 3 + \sqrt{3}, \quad 3 - \sqrt{3}, \quad 1 - \sqrt{3}]$ and symbol $\varphi_2(x)(1 + \sqrt{3} + (1 - \sqrt{3}))\,e^{-ix})/2$.

Considering transfer grid operators for multigrid methods, a question now seems to be natural: "Is the orthogonality condition (20) necessary or the moment condition (19) is sufficient?" The answer is that the orthogonality condition (20) is not necessary. Some motivations were firstly given in [4]. Indeed, in the MRA we would have $V_0 = V_1 \oplus W_1$, while for multigrid methods we would have $\Omega = \text{Range}\{P\} \oplus \text{NullSpace}\{R\}$ where Ω^h is the fine grid [5]. The error e in Ω^h can be decomposed as $e = s + t$, with $s \in \text{Range}\{P\}$ and $t \in \text{NullSpace}\{P^T\}$ (we consider the Galerkin condition $R = P^T$). We note that $CGCs = 0$, while $CGCt = t$. Therefore, if after the pre-smoothing step $t = 0$, then the TGM converges in one iteration. Clearly this is a too strong condition. However $\text{Range}\{P\}$ is spanned by smooth functions while $\text{NullSpace}\{P^T\}$ is spanned by oscillating functions, thus if we have a "good" smoother $t \approx 0$. Consequently, to define a good grid transfer operator we are interested only to the scaling function and we do not use the wavelet since the smoother has already reduced the error in the high frequencies. Eventually, the multigrid methods is an iterative method, hence it is not necessary to have convergence in one iteration but the solution can be substantially improved iterating. These considerations show that for multigrid methods the orthogonality condition (20) is not necessary. The only necessary condition concerns moments in (19), that is equivalent to the HF and to the factorization $\varphi_m \nu_m$.

Concluding, for Toeplitz linear system relations between wavelets and multigrid methods was already investigated in [7]. In such paper the authors proved the TGM optimality using Cohen, Daubechies and Feauveau (CDF) 9/7 biorthogonal wavelets for generating functions having a zero of order 4 in the origin. Thanks to the previous comments this is expected since CDF 9/7 have both moments of order 4. However the interesting fact is that the proof in [7] is done directly in the Toeplitz class and not in the algebra case. Moreover, computing a class of compactly supported biorthogonal wavelet systems GCDF with specified vanishing moments for scaling functions, the authors show numerically that for moments of order k the TGM is optimal for generating functions having a zero of order at most $2k$. This is exactly the condition (9a) since the symbols have the form $H(x) = \varphi(x)\nu(x)$.

7   Numerical Experiments

For the sake of simplicity, we consider some tests in the 1D case. The results in the multidimensional case are very similar. The numerical experiments are done using Matlab 7.0. We fix the following parameters. The MGMs are stopped when the relative l_2 norm of residual is lower than 10^{-9}. The coarse matrices are defined using the Galerkin approach. The coarser problem has dimensions 7×7. These means that $A_{n^{(i)}} = T_{n^{(i)}}(f_i) + \text{low rank}$, for $i = 1, \ldots, l$, where the f_i are defined according to (12). The right hand side is obtained

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from the exact solution $x_j = j/n$, $j = 1, \ldots, n$. The pre-smoother and the post-smoother are one step of relaxed Richardson with relaxation parameter equal to $1.5/\|f_i\|_\infty$ and $1/\|f_i\|_\infty$ according to [1].

In Section 4, Proposition 5 gives a validation of the TGM conditions (11) for PDEs. Here, in the first test problem, we give a numerical validation of the conditions (11) in the case of discretized integral problems. We show that sometimes the choice $r \neq p$ can be very useful also for the MGM for Toeplitz matrices. In the second test problem, we show that the V-cycle optimality result for Toeplitz matrices in Proposition 7 can be applied also to non-constant coefficients PDEs.

7.1 An integral problem

We consider the discretization by the rectangle quadrature formula of the following Fredholm operator of first kind:

$$g(x) = \int_\mathbb{R} t(x - \theta)\, f(\theta)\, \mathrm{d}\theta, \qquad x, \theta \in \mathbb{R},$$

where f is the input object, t is the integral kernel of the operator, also called point spread function (PSF) and g is the observed object. In the discrete case, when zero Dirichlet BCs are used and the PSF is shift invariant, the above approximation gives rise to the system $A_n\mathbf{f} = \mathbf{g}$, where $A_n = T_n(z)$ with $z(\pi) = 0$ and positive elsewhere. According to the analysis in Section 4, the generating function of a grid transfer operator of order s will be μ_s at the finer level and ϕ_s at the lower level. We denote by δ_r and δ_p the order of the zero of the restriction and of the prolongation, respectively.

First of all, we give a numerical evidence of the relevance of conditions (11). We consider $z(x) = (2 + 2\cos(x))^3$, which has a zero in π of order 6. From Proposition 3 $\delta_r = \delta_p = 2$ is not enough for an optimal TGM, while it is necessary to set $\delta_r = \delta_p = 4$. However, if we allow $\delta_r \neq \delta_p$ then, from conditions (11), the optimality of the TGM is guaranteed with $\delta_r = 2$ and $\delta_p = 4$. This is confirmed by the numerical results in Table 2. In such table we report the number of iterations required by the TGM for converging for different orders of the grid transfer operators, when increasing the problem size.

In real applications usually more than two-grids are used. The TGM optimality conditions are not enough for the V-cycle, but they give good estimations for the W-cycle [20]. Defining $z(x) = (2 + 2\cos(x))^2$, which has a zero in π of order 4, also the choice $\delta_r = \delta_p = 2$ gives an optimal TGM. However, in Table 3 we see that for the W-cycle there is a large reduction in the iteration number for $\delta_r = 2$ and $\delta_p = 4$, with respect to $\delta_r = \delta_p = 2$. Furthermore using $\delta_r = \delta_p = 4$

Table 2
TGM iteration numbers varying the order of the grid transfer operators and the problem size n for the integral problem $z(x) = (2 + 2\cos(x))^3$.

n	$\delta_r = 2$ $\delta_p = 2$	$\delta_r = 2$ $\delta_p = 4$	$\delta_r = 4$ $\delta_p = 4$
15	219	65	51
31	607	72	52
63	1501	76	51
127	> 2000	77	50
255	> 2000	78	49

implies a negligible reduction of the iteration number, with respect to the choice $\delta_r = 2$ and $\delta_p = 4$.

From a computational point of view, we should investigate the structure of the coefficient matrices at each level for the previous choices of the grid transfer operators. Indeed, at each level the main computational cost is related to the matrix vector product with the matrices $A_{n^{(i)}}$, for $i = 0, \ldots, l$. In the last example with $z(x) = (2 + 2\cos(x))^2$, we use the following grid transfer operators of order $2s$: $r_0 = p_0 = (2 - 2\cos(x))^s$ and $r_i = p_i = (2 + 2\cos(x))^s$ for $i = 1, \ldots, l - 1$. For $\delta_r = \delta_p = 2$ we have $A_{n^{(i)}} = T_n^{(i)}(\tilde{z})$, for $i = 1, \ldots, l$, where $\tilde{z}(x) = (2 - 2\cos(x))^2$. For $\delta_r = 2$ and $\delta_p = 4$ we have $A_{n^{(i)}} = 2^i T_n^{(i)}(\tilde{z}) + c_i e_1 e_1^T + c_i e_n e_n^T$, where e_j is the jth vector of the canonical basis and $c_i = [T_n^{(i)}(r_i) A_{n^{(i)}} T_n^{(i)}(p_i)]_{2,2} - 2^i \cdot 6$. Therefore, the matrix vector product has about the same computational cost for both choices. On the other hand, for $\delta_r = \delta_p = 4$ the matrix $A_{n^{(i)}}$ is Toeplitz plus a 4 rank correction and moreover the bandwidth of the Toeplitz part is not longer 5, but it becomes 7. Obviously, this fact increases the complexity and the computational cost of the matrix vector product. Moreover, for using the Gauss-Seidel smoother with a coloring strategy we need 4 colors instead of 3 colors as for $T_n(z)$, losing a degree of parallelism. The previous considerations are enhanced in the multidimensional case. Indeed in the two dimensional case the bandwidth of each block moves from 5 to 7, and also the block bandwidth moves from 5 to 7.

For preserving the Toeplitz structure at each level, a different cutting matrix proposed in [2] can be used. The main idea is in the changing of the coarse problems size in order to neglect in some way the boundary effects that give the low rank corrections. However, in this way we lose some information and the iteration number slightly increases even if the general (optimal) behavior is preserved in the numerical experimentations.

Table 3
W-cycle iteration numbers varying the order of the grid transfer operators and the problem size n for the integral problem $z(x) = (2 + 2\cos(x))^2$.

n	$\delta_r = 2$ $\delta_p = 2$	$\delta_r = 2$ $\delta_p = 4$	$\delta_r = 4$ $\delta_p = 4$
31	25	23	22
63	32	23	21
127	35	23	21
255	37	23	20
511	37	23	20

7.2 A differential equation with nonconstant coefficients

We consider the following equation

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(a(x)\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}u(x)\right) = g(x), & x \in (0,1), \\ u(0) = u(1) = 0 \end{cases} \tag{21}$$

with nonconstant $a(x)$ and order $m = 4$.

In this subsection we consider the V-cycle, that is cheaper than the W-cycle in vector and parallel implementations. Therefore we need a more powerful smoother and we replace the weighted Richardson with Gauss-Seidel. The generating function of the grid transfer operator is the same for each coarse problem. We test several combinations of ϕ_k, $k = 2, 4$ and the cubic interpolation g_c.

We note that in practical implementations it is usually required that (7) is satisfied strictly. This condition in terms of generating functions is equivalent to require that $c = 0$ in (11a): the latter is a generalization of a similar condition already numerically observed in [2] for $r = p$ and when considering the V-cycle. Indeed, from Table 4, we see that the choice (ϕ_2, ϕ_2) for the couple (restriction, prolongation) is not effective. We recall that ϕ_2 is the linear interpolation. For obtaining an effective V-cycle it is enough to increase only the order of the prolongation (i.e., $\gamma_r = 2$ and $\gamma_p = 4$). However, more stable results can be obtained for $\gamma_r = 4$ and $\gamma_p = 4$ according to Proposition 7.

Eventually, we compare the two prolongations generated by ϕ_4 and g_c. In terms of number of iterations the function g_c has to be preferred with respect to ϕ_4 according to Remark 6. On the other hand, ϕ_4 has a 5-point stencil while g_c has

Table 4

V-cycle iteration numbers varying problem size n and $a(x)$ for the differential problem (21) (g_c = cubic interpolation).

restriction	ϕ_2	ϕ_2	ϕ_2	ϕ_4	ϕ_4	ϕ_2	ϕ_2	ϕ_2	ϕ_4	ϕ_4
prolongation	ϕ_2	ϕ_4	g_c	ϕ_4	g_c	ϕ_2	ϕ_4	g_c	ϕ_4	g_c
n	$a(x) = \mathrm{e}^x$					$a(x) = (x - 0.5)^2$				
15	14	9	9	7	7	15	10	10	9	9
31	32	11	13	10	9	33	13	17	10	11
63	60	17	15	14	9	61	17	24	13	11
127	98	27	20	18	12	101	26	27	17	13
255	151	38	27	22	16	155	35	29	20	16
511	215	48	34	26	20	221	44	36	24	19
1023	276	57	44	29	22	284	53	46	27	22

a 7-point stencil. This means that the choice (ϕ_2, g_c) leads to coarse matrices having a bandwidth equal to 7 like the choice (ϕ_4, ϕ_4), while (ϕ_4, g_c) leads to a bandwidth equal to 9. Therefore, the right comparison should be between (ϕ_2, g_c) and (ϕ_4, ϕ_4). From Table 4, it is evident that the second one has to be preferred according to Remark 7. Moreover, considering the discretization near the boundary, ϕ_k requires less boundary points than g_c.

8   Conclusions

Considering elliptic PDEs with constant coefficients, we have shown the equivalence between the LFA and the analysis based on the zeros of the generating functions of Toeplitz matrices. This equivalence has two implications. The first one is that the techniques used for Toeplitz and Circulant matrices allow to extend the LFA also to non-differential problems (e.g., integral problems). The second one is that it suggests to choose the restriction different from the prolongation also for MGMs for Toeplitz linear systems. The generalization of the MGM for Toeplitz matrices proposed in Section 4 replaces the Galerkin conditions with the following:

(1) $A_{n/2} = R_n(r)A_n P_n(p)$
(2) r not necessary equal to p, but such that $p \geq 0$, $r \geq 0$ and both even or odd (such that $A_{n/2}$ is again positive definite).

We have given a class of grid transfer operator with minimal support for a fixed HF and the geometrical interpretation of the operator with HF = 4. Such

class is related to the symbol of B-spline, giving the possibility to discuss some useful and suggestive relations between wavelets and multigrid methods.

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